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Holography, Duality and Higher-Spin Theories

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Abstract

I review recent work on the holographic relation between higher-spin theories in Anti-de Sitter spaces and conformal field theories. I present the main results of studies concerning the higher-spin holographic dual of the three-dimensional $O(N)$ vector model. I discuss the special role played by certain double-trace deformations in Conformal Field Theories that have higher-spin holographic duals. Using the canonical formulation I show that duality transformations in a $U(1)$ gauge theory on AdS_4 induce boundary double-trace deformations. I argue that a similar effect takes place in the holography of linearized higher-spin theories on AdS_4.

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Notes taken by K. Alkalaev.

1 Introduction

Consistently interacting higher-spin (HS) gauge theories exist on Anti-de Sitter spaces (see [1, 2] for recent reviews and extensive literature on higher-spin gauge theories). It is then natural to ask whether HS theories have interesting holographic duals. In this lecture I review recent work on the holographic relation between HS theories and conformal field theories (CFTs). After some general remarks concerning the relevance of HS theories to free CFTs, I present the main results in studies of the three-dimensional critical $O(N)$ vector model that has been suggested to realize the holographic dual of a HS theory on AdS_4. Furthermore, I discuss the special role the so-called double-trace deformations seem to play in the dynamics of CFTs that have holographic HS duals. I particular, I show that duality transformations of a $U(1)$ gauge theory on AdS_4 induce boundary double-trace deformations. I argue that a similar effect takes place in the implementation of holography to linearized HS theories on AdS_4.

2 Higher-spin currents and the operator spectrum of Conformal Field Theories

The operator spectrum of d-dimensional Conformal Field Theories (CFTs) consists of an infinite set of modules each containing one quasi-primary operator $\Phi(0)$ that is annihilated by the generator $\hat{K}_\mu$ of special conformal transformations [3]

$$\hat{K}_\mu \Phi(0) = 0 \; , \tag{2.1}$$

as well as an infinite number of descendants that are essentially the derivatives of $\Phi(0)$. Thus, quasi-primary operators carry irreps of the conformal group $SO(d,2)$ labeled only by their spin s and scaling dimension Δ. As a consequence, the 2- and 3-pt functions of quasi-primary operators are determined up to a number of constant parameters (see for example [4]).

Perhaps the most important property of quasi-primary operators is that they form an algebra under the (associative) operator product expansion (OPE) [5]. Given two such operators $A(x)$ and $B(x)$ we may expand their product in a (infinite) set of quasi-primary operators $\{Q\}$ as

$$A(x)B(0) = \sum_{\{Q\}} C(x,\partial)Q(0) \; . \tag{2.2}$$

The coefficients $C(x,\partial)$ are fully determined in terms of the spins, dimensions and 3-pt functions of the operators involved in the OPE. Knowledge of the OPE is potentially sufficient to determine all correlation functions of quasi-primary operators in CFTs.

In specific models all quasi-primary operators (with the exception of the "singletons"), are composite operators and determining their precise list, spins and dimensions is a hard task which is done in practice by studying 4-pt functions. For example, consider a scalar quasi-primary operator $\Phi(x)$ with dimension Δ. The OPE enables us to write

$$\begin{aligned}\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle &= \sum_{\{Q\}} C(x_{12},\partial_2)C(x_{34},\partial_4)\langle Q(x_2)Q(x_4)\rangle \\ &= = \frac{1}{(x_{12}^2x_{34}^2)^\Delta}\sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, H_{\Delta_s,s}(v,Y) \\ &= \frac{1}{(x_{12}^2x_{34}^2)^\Delta}\sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, v^{\frac{\Delta_s-s}{2}}Y^s\Big[1+O(v,Y)\Big]\,, \qquad (2.3)\end{aligned}$$

where we have used the standard harmonic ratios

$$v=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\qquad u=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}\,,\qquad Y=1-\frac{v}{u},\qquad x_{ij}^2=|x_i-x_j|^2\,. \qquad (2.4)$$

The coefficients $g_{\Delta_s,s}$ are the "couplings". The expressions $H_{\Delta_s,s}(u,Y)$ are complicated but explicitly known [6] functions that give the contribution of an operator with spin s and dimension Δ to the OPE. The terms $v^{(\Delta_s-s)/2}Y^s$ correspond to the leading short-distance behavior of the 4-pt function for $x_{12}^2\,,x_{34}^2\to 0$.

In the simplest case of a massless free CFT the set $\{Q\}$ includes, among others, all symmetric traceless quasi-primary operators with spin s and dimension Δ_s. It is a general result in CFT that if the above operators are also conserved, then their dimensions are given by [1]

$$\Delta_s=d-2+s\,. \qquad (2.5)$$

Therefore, whenever higher-spin conserved currents reside in the list of quasi-primary operators each one contributes in (2.3) a term of the form

$$v^{\frac{d-2}{2}}Y^s\,. \qquad (2.6)$$

Of course, irreps with canonical scaling dimension (2.5) are nothing but the massless UIRs of the $d+1$-dimensional Anti-de Sitter group.

Suppose now that one is interested in the holographic description of the 4-pt function (2.3). For that, we notice that from a classical action on Anti-de Sitter we get non-trivial boundary correlators by studying tree-level bulk-to-boundary graphs [8].

[1]The converse is not always true [7], therefore even if one finds terms such as (2.6) in the OPE one should be careful in interpreting them.

Hence, when the boundary 4-pt function contains terms like (2.6) it is necessary to consider bulk massless currents. On the other hand, the energy-momentum tensor always appears in the OPE of a 4-pt function, it has dimension d and spin-2. Its dimension remains canonical. Its contribution to the 4-pt function is given by the (rather complicated) function

$$
\begin{aligned}
H_{e.m.}(v,Y) &= vF_1(Y)[1+O(v,Y)]\,, && (2.7)\\
F_1(Y) &= \frac{4Y^2-8Y}{Y^3}+\frac{4(-6+6Y-Y^2)}{Y^3}\ln(1-Y)\longrightarrow Y^2+... && (2.8)
\end{aligned}
$$

A concrete realization of the ideas above is provided by explicit calculations in $\mathcal{N}=4$ SYM$_4$ via the AdS/CFT correspondence [9]. Consider the 4-pt function of the so-called lower dimension chiral primary operators (CPOs) of $\mathcal{N}=4$, which are scalar operators with dimension 2. In the free field theory limit we have

$$
\begin{aligned}
&\frac{\delta^{I_1I_2}\delta^{I_3I_4}}{400}\langle Q^{I_1}(x_1)...Q^{I_4}(x_4)\rangle_{free}=\\
&\qquad=\frac{1}{(x_{12}^2x_{34}^2)^2}\Big(1+\frac{1}{20}v^2+\frac{1}{20}v^2(1-Y)^{-2}\\
&\qquad+\frac{4}{N^2}(\frac{1}{6}[v+v(1-Y)^{-1}]+\frac{1}{60}v^2(1-Y)^{-1})\Big)\\
&\qquad=\frac{1}{(x_{12}^2x_{34}^2)^2}\Big(...+\frac{4}{6N^2}\sum_{l=2}^{\infty}vY^l+...\Big)\,. \qquad (2.9)
\end{aligned}
$$

In the last line of (2.9) we see the contribution of an infinite set of higher-spin conserved currents in the *connected* part of the correlator. The leading contribution comes from the energy-momentum tensor. One can identify the contributions from all the higher-spin currents and even calculate their "couplings", after some hard work that involves the subtraction of the descendants of each and every current. The perturbative corrections to the connected part of the free result above have the form

$$
[connected]=\frac{1}{N^2}[connected]_{free}+\frac{1}{N^2}g_{YM}^2N\,F(v,Y)\,, \qquad (2.10)
$$

where

$$
F(v,Y)\sim\sum_l v\,Y^l\ln v+... \qquad (2.11)
$$

The above terms can be attributed to an infinite set of "nearly conserved" higher-spin currents i.e. quasi-primary operators whose scaling dimensions have being shifted from their canonical value as [9]

$$
\Delta_{HS}\longrightarrow\Delta_{HS}+\gamma=2+s+(g_{YM}^2N)\eta_s+\cdots\,. \qquad (2.12)
$$

By the operator/state correspondence in CFTs, we may view the above effect as a small deformation of the energy spectrum of the theory. From the AdS side this deformation should correspond to a Higgs-like effect by which the initially massless higher-spin currents acquire masses [10, 11].

In the context of the AdS_5/CFT_4 correspondence, one can calculate the same 4-pt function using IIB supergravity [12]. The result is highly non-trivial and looks like

$$[connected] = \frac{1}{N^2}\frac{1}{(x_{12}^2x_{34}^2)^2}\Big[vF_1(Y) + O(v^2, Y)\Big]\,. \tag{2.13}$$

It is an astonishing fact that the expansion of such a non-trivial function reveals the presence of *only* the energy momentum tensor and the absence of *all* higher-spin currents. This shows how far away supergravity is from a holographic description of perturbative CFTs. This shows also the necessity to consider HS gauge theories if we wish to describe holographically perturbative CFTs.

3 Holography of the critical three-dimensional $O(N)$ vector model

A concrete proposal for the holographic correspondence between a CFT and a HS gauge theory was made in [13]. It was there suggested that the critical three-dimensional $O(N)$ vector model is the holographic dual of the simplest HS gauge theory on AdS_4, a theory that contains bosonic symmetric traceless even-rank tensors. The elementary fields of the (Euclidean) three-dimensional $O(N)$ vector model are the scalars

$$\Phi^a(x)\,, \quad a = 1, 2, .., N\,, \tag{3.14}$$

constrained by

$$\Phi^a(x)\Phi^a(x) = \frac{1}{g}\,. \tag{3.15}$$

The model approaches a free field theory for $g \to 0$. To calculate the partition function in the presence of sources $J^a(x)$ it is convenient to introduce the Lagrange multiplier field $G(x)$ as

$$Z[J^a] = \int (\mathcal{D}\Phi^a)(\mathcal{D}G) e^{-\frac{1}{2}\int \Phi^a(-\partial^2)\Phi^a + \frac{i}{2}\int G(\Phi^a\Phi^a - \frac{1}{g}) + \int J^a\Phi_a}\,, \tag{3.16}$$

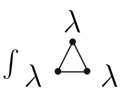
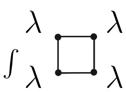
From (3.16) we see that is is natural to consider the effective coupling $\hat{g} = gN$ which for large-N may be adjusted to remain $O(1)$ as $g \to 0$. Integrating out the $\Phi^a s$ and setting $G(x) = G_0 + \lambda(x)/\sqrt{N}$ we obtain the (renormalizable) $1/N$ expansion as

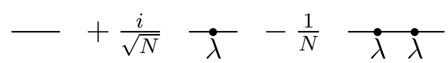
$$\frac{Z[J^a]}{Z_0} = \int (\mathcal{D}\lambda) e^{-\frac{N}{2}[Tr(\ln(1-\frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})) + \frac{i}{\sqrt{N}}\frac{\lambda}{\hat{g}}]} e^{\frac{1}{2}\int J^a \frac{1}{-\partial^2}(1-\frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})^{-1} J^a}\ , \tag{3.17}$$

where Z_0 depends on G_0. The critical theory is obtained for $G_0 = 0$ and the critical coupling is determined by the gap equation

$$\frac{1}{\hat{g}_*} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}\,. \tag{3.18}$$

The resulting generating functional takes the form

$$\frac{Z_*[J^a]}{Z_0} = \int (\mathcal{D}\lambda) e^{-\frac{1}{2}\int \lambda K \lambda - \frac{i}{3!\sqrt{N}}\int}$$  $$- \frac{1}{8N}\int$$  $$+ \cdots$$

$$\times\ e^{\frac{1}{2}\int J^a [}$$  $$+ \cdots] J^a\ . \tag{3.19}$$

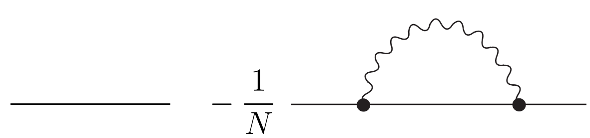
The basic propagator of the $\Phi^a(x)$s is

$$\Delta(x) = \frac{1}{4\pi}\frac{1}{(x^2)^{1/2}} \quad \sim \tag{3.20}$$

The operator K and its inverse are then found to be

$$K = \frac{\Delta^2}{2}\,, \quad K^{-1} = -\frac{16}{\pi^2}\frac{1}{x^4} \quad \sim \quad - \tag{3.21}$$

Now we can calculate all n-pt functions of Φ^a. For example, the 2-pt function is given by

$$\langle \Phi^a \Phi^b \rangle = \delta^{ab}[$$  $$+\cdots]$$

$$= \delta^{ab}\frac{1}{(x^2)^{1/2}}[1 - \eta_1 \ln x^2 + ...]\,, \qquad \eta_1 = \frac{1}{N}\frac{4}{3\pi^2}\,. \tag{3.22}$$

We notice that the elementary fields have acquired an anomalous dimension η_1. Ideally, a holographic description of the $O(N)$ vector model should reproduce this result from a bulk calculation, however, such a calculation is still elusive. On the other hand, bulk fields would give the correlation functions of composite boundary operators. The generating functional for one such operator may be obtained if we consider a external source for the fluctuations of the auxiliary field as

$$
\frac{Z[J^a]}{Z_0} \to Z[A]
$$

$$
= \int (\mathcal{D}\lambda) e^{-\frac{1}{2}\int \lambda K \lambda \, - \, \frac{i}{3!\sqrt{N}} \int \lambda \overset{\lambda}{\triangle} \lambda \, - \, \frac{1}{8N} \int {}^{\lambda}_{\lambda} \square {}^{\lambda}_{\lambda} \, + \cdots + \int A\lambda} \, . \tag{3.23}
$$

This can be viewed as the generating functional $e^{\hat{W}[A]}$ for a conformal scalar operator λ with a dimension $\Delta = 2 + O(1/N)$. To be precise, since

$$
\int (\mathcal{D}\lambda) e^{-\frac{1}{2}\int \lambda K \lambda + \int A\lambda} = e^{\frac{1}{2}\int A \Pi \lambda} \, , \tag{3.24}
$$

and Π would give a non-positive 2-pt function, we should actually consider

$$
W[A] = \hat{W}[iA] \, . \tag{3.25}
$$

The proposal of [13] may then be concretely presented as

$$
e^{W[A]} \equiv \int_{AdS_4} (\mathcal{D}\Phi) e^{-I_{HS}(\Phi)} \, , \tag{3.26}
$$

where

$$
I_{HS}(\Phi) = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} \Big(\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2 + ... \Big) \, , \tag{3.27}
$$

with $m^2 = -2$ that corresponds to conformally coupled scalar on AdS_4. We see that the problem in hand naturally asks for a "bottom-up" approach, namely, use the full knowledge of the boundary effective action in order to calculate the bulk path integral. Indeed, in principle we should be able to have control of the fully quantized bulk theory, since bulk quantum corrections would correspond to the $1/N$ corrections of a renormalizable boundary theory. For the time being, however, one can be content if knowledge of the boundary generating functional for composite operators can help her calculate the elusive classical bulk action for a HS gauge theory.

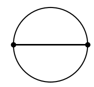
Let me briefly describe now, how this "lifting program" works [14]. A possible form of the bulk HS action is

$$
\begin{aligned}
I_{HS}(\Phi) \quad = \quad & \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} \Big(\frac{1}{2}(\partial\Phi)^2 - \Phi^2 + \frac{g_3}{3!}\Phi^3 + \frac{g_4}{4!}\Phi^4 \\
& + \sum_{s=2}^{\infty} \mathcal{G}_s h^{\{\mu_1 \ldots \mu_s\}} \Phi \, \partial_{\{\mu_1} \ldots \partial_{\mu_s\}} \Phi + \cdots \Big) \, ,
\end{aligned} \tag{3.28}
$$

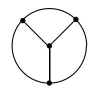
where $\mathcal{G}_s$ denote the couplings of the conformally coupled scalar to the higher-spin gauge fields $h^{\{\mu_1,..,\mu_s\}}$ that can be taken to be totally symmetric traceless and conserved

tensors. From (3.28), by the standard holographic procedure involving the evaluation of the on-shell bulk action with specified boundary conditions, we obtain schematically:

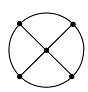
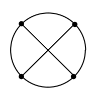
– 2-pt function –

$$\langle\lambda\lambda\rangle \quad \sim$$

$$\tag{3.29}$$

– 3-pt function –

$$\langle\lambda\lambda\lambda\rangle \quad \sim \quad g_3$$

$$\tag{3.30}$$

The 4-pt function depends on g_4, g_3^2 and on *all* the higher-spin couplings $\mathcal{G}_s$. It is technically not impossible to write down the bulk tree graphs that involve the exchange of higher-spin currents [15]. Moreover, bulk gauge invariance should, in principle, be sufficient to determine *all* the $\mathcal{G}_s$ in terms of only one of them i.e. in terms of the coupling $\mathcal{G}_2$ of the energy momentum tensor. Schematically we have

$$\langle\lambda\lambda\lambda\lambda\rangle \quad \sim \quad g_4$$

$$+ \quad g_3^2\Big[$$

$$+\,\text{crossed}\Big] + \sum_s \mathcal{G}_s\Big[\cdots\Big] \tag{3.31}$$

The above must be compared with the corresponding result obtained from the $W[A]$. This would allow us to fix the scalings of various coefficients as (we set the AdS radius to 1)

$$\frac{1}{2\kappa_4^2} \sim N \ , \quad g_3\,,g_4 \sim O(1) \ , \quad \mathcal{G}_s^2 \sim O(\frac{1}{N}) \ \text{ iff } \ \langle h_s\, h_s\rangle \sim O(1) \ . \tag{3.32}$$

The first result obtained this way was [14]

$$g_3 = 0 \ . \tag{3.33}$$

This has been confirmed by a direct calculation using the Vasiliev equations on AdS$_4$ [16]. Further results have been reported in [15], however much more needs to be done to get a satisfactory understanding of the bulk HS action. Another open issue is to reproduce, from bulk loops, the known boundary anomalous dimension of the operator λ. Finally, it remains to be understood [2] how the Higgs mechanism for the bulk HS fields gives rise to the known $1/N$ corrections of the anomalous dimensions of the corresponding boundary higher-spin operators.

[2] See [17] for a recent work on that issue.

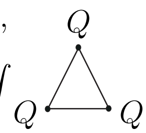
4 The role of boundary double-trace transformations and the first trace of duality

In our study of the critical $O(N)$ vector model we have started with an elementary field Φ^a with dimension $\Delta = 1/2 + O(1/N)$ and obtained a composite operator λ with $\Delta_\lambda = 2 + O(1/N)$. It follows that we are dealing with an interacting CFT even for $N \to \infty$, since the free CFT would have had a composite operator like $\frac{1}{\sqrt{N}}\Phi^a\Phi^a$ with $\Delta_{\Phi^2} = 1$. It is then natural to ask where is the free theory? To find it we consider the Legendre transform of $W[A]$ as

$$W[A] + \int AQ = \Gamma[Q]\,, \tag{4.34}$$

$$\Gamma[Q] = \Gamma_0[Q] + \frac{1}{N}\Gamma_1[Q] + \ldots\,, \tag{4.35}$$

$$\Gamma_0[Q] = \frac{1}{2}\int Q\,K\,Q + \frac{1}{3!\sqrt{N}}\int$$



$$+\cdots\,. \tag{4.36}$$

The generating functional for the correlation functions of the free field $\frac{1}{\sqrt{N}}\Phi^a\Phi^a$ with dimension $\Delta = 1$ is $\Gamma_0[Q]$. Therefore, we face here the holographic description of a free field theory!

The theory described by $\Gamma[Q]$ has imaginary couplings and anomalous dimensions below the unitarity bounds [18]. Nevertheless, *both* the theories described by $W[A]$ and $\Gamma[Q]$ should be the holographic duals of a *unique* HS bulk theory. Moreover, it turns out that these two theories are related to each other by an underlying dynamics that appears to be generic in non-trivial models of three-dimensional CFT. We propose that this particular underlying dynamics is a kind of duality. This is motivated by the fact that for $N \to \infty$ the spectrum of free field theory and the spectrum of theory $W[A]$ are *almost* the same. Indeed, the $W[A]$ theory contains conserved currents for $N \to \infty$. The *only* difference between the two theories at leading-N is the interchange of the two scalar operators corresponding to the two following UIRs of SO(3,2)

$$D(1,0) \longleftrightarrow D(2,0)\,. \tag{4.37}$$

These UIRs are *equivalent* as they are related by Weyl reflections and have the same Casimirs.

The *duality* (4.37) is induced by a particular type of dynamics usually referred to as *double-trace deformations*.[3] To show the essence of our proposal, consider an operator

[3] We keep this terminology despite the fact that there are no traces taken here.

$Q(x)$ with a dimension $\Delta = 1$ i.e. an operator in free field theory. Then $Q^2(x)$ is a relevant deformation of a theory and we can consider the deformed 2-pt function as

$$\begin{aligned}&\langle Q(x_1)Q(x_2)\, e^{\frac{f}{2}\int Q^2(x)}\rangle = \langle Q(x_1)Q(x_2)\rangle_f \\ &= \langle Q(x_1)Q(x_2)\rangle_0 + \frac{f}{2}\int d^3x \langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 + \dots \,.\end{aligned} \tag{4.38}$$

We now make a large-N factorization assumption such that, for example,

$$\frac{1}{2}\langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 \simeq \langle Q(x_1)Q(x)\rangle_0 \langle Q(x_2)Q(x)\rangle_0 + O\left(\frac{1}{N}\right)\,, \tag{4.39}$$

and similarly for all correlators that appear in (4.38). We then obtain

$$\begin{aligned}\langle Q(x_1)Q(x_2)\rangle \;&=\; \langle Q(x_1)Q(x_2)\rangle_0 \\ &\quad + f\int d^3x \langle Q(x_1)Q(x)\rangle_0 \langle Q(x_2)Q(x)\rangle + O\left(\frac{1}{N}\right)\,.\end{aligned} \tag{4.40}$$

In momentum space this looks like

$$Q_f(p) = \frac{Q_0(p)}{1 - fQ_0(p)}\,, \quad Q_0(p) \simeq \frac{1}{p}\,. \tag{4.41}$$

In the infrared, i.e. for small momenta $|p| \ll f$, we find

$$f^2 Q_f(p) = -\frac{f}{1 - \dfrac{1}{fQ_0(p)}} \simeq -f - Q_0^{-1}(p) + \dots\,. \tag{4.42}$$

If we drop the non-conformal constant f term on the r.h.s. of (4.42) we obtain the 2-pt function of an operator with dimension $\Delta_f = 2$. We see that the UV dimension $\Delta_0 = 1$ has changed to the IR dimension $\Delta_f = 2$ and that this change is induced by the double-trace deformation.

The above dynamics must be seen in AdS_4. The on-shell bulk action of a conformally coupled scalar, using the standard Poincaré coordinates, is

$$I_\varepsilon = -\frac{1}{2}\frac{1}{\varepsilon^2}\int d^3x\, \Phi(\bar{x};\varepsilon)\partial_r \Phi(\bar{x};r)\Big|_{r=\varepsilon\ll 1}\,. \tag{4.43}$$

To evaluate it we need to solve the Dirichlet problem

$$(\nabla^2 - 2)\Phi(\bar{x};r) = 0\,, \quad \Phi(\bar{x};r=\infty) = 0\,, \quad \Phi(\bar{x};\varepsilon) = \Phi(\bar{x};\varepsilon)\,. \tag{4.44}$$

$$\Phi(\bar{x};r) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{x}\bar{p}}\Phi(\bar{p};\varepsilon)\,\frac{r}{\varepsilon}\, e^{-|p|(r-\varepsilon)}\,. \tag{4.45}$$

One way to proceed is via the Dirichlet-to-Neumann map [19] that relates the boundary value of a field in a certain manifold $\mathcal{M}$ to its normal derivative at the boundary, e.g.

$$\Phi(x)\Big|_{x\in\partial\mathcal{M}} = f(\bar{x}) \; ; \quad \hat{\Lambda} f = n^\mu \partial_\mu \Phi(x)\Big|_{x\in\partial\mathcal{M}} \, , \tag{4.46}$$

where n^μ is the normal to the boundary vector. Knowledge of the map $\hat{\Lambda}$ allows the reconstruction of the bulk metric. For the conformally coupled scalar we have the remarkably simple expression

$$\partial_r \Phi(\bar{p}; r)\Big|_{r=\varepsilon} = (\frac{1}{\varepsilon} - |p|)\Phi(\bar{p};\varepsilon) \, . \tag{4.47}$$

The terms in parenthesis on the r.h.s. may be viewed as a *generalized Dirichlet-to-Neumann map* since we have taken the boundary to be at $r = \epsilon$. This map may be identified with the r.h.s. of the expansion (4.42) is we set $f = 1/\epsilon$, such that

$$\frac{1}{\epsilon^2}\left[\hat{\Lambda}_\epsilon(p)\right]^{-1} \sim f^2 Q_f(p) = f^2 \frac{Q_0(p)}{1 - f Q_0(p)} \, . \tag{4.48}$$

We then see that the inversion of the generalized Dirichlet-to-Neumann map for a conformally coupled scalar corresponds to the resummation induced by a double-trace deformation on the free boundary 2-pt function. Notice that the limit $\epsilon \to 0$ drives the boundary theory in the IR.

Let us finally summarize in a pictorial way some of the salient features of the two types of holography discussed above. In Fig.1 we sketch the standard holographic picture for the correspondence of IIB string-SUGRA/$\mathcal{N} = 4$ SYM. In Fig.2 we sketch what we have learned so far regarding the holography of the HS gauge theory on AdS$_4$.

5 Double-trace deformations and the duality of linearized HS gauge theories

In the previous section we emphasized the special role played by the boundary double-trace deformation

$$\frac{f}{2}\int d^3x \, Q(x)Q(x) \, , \tag{5.49}$$

for theories with bulk conformally invariant scalars. This raises the possibility that boundary deformations of the form

$$\frac{f_s}{2}\int d^3x \, h^{(s)} h_{(s)} \, , \tag{5.50}$$

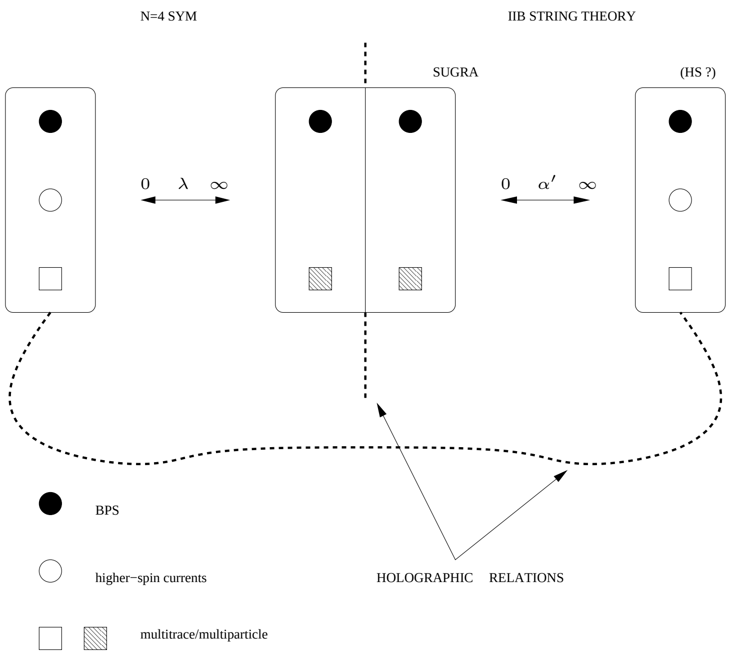


Figure 1: Type 1 holographic correspondence: $\mathcal{N} = 4$ SYM/IIB string theory.

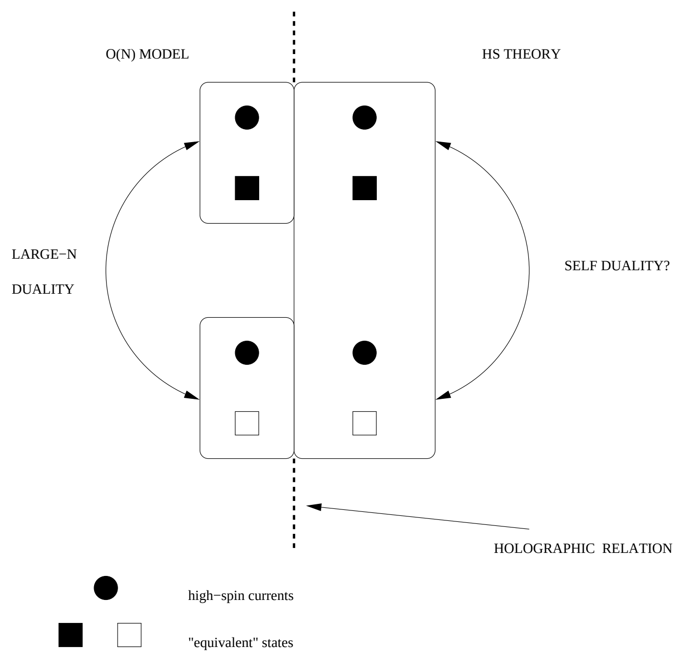


Figure 2: Type 2 holographic correspondence: O(N) vector model/HS on AdS_4.

where $h^{(s)}$ denote symmetric traceless and conserved currents, may play a crucial role in the holography of HS gauge theories. Such deformations are of course *irrelevant* for all $s \geq 1$, nevertheless in many cases they lead to well-defined UV fixed points. Important examples are the three-dimensional Gross-Neveu and Thirring models in their large-N limits [20]. We should emphasize that the large-N limit is absolutely crucial for the well-defined and non-trivial nature of the above results.

For example, the dynamics underlying the Gross-Neveu model is, in a sense, just the opposite of the dynamics described in the previous section and is again connected with the equivalence or "duality" between the irreps $D(1,0)$ and $D(2,0)$ [21]. However, we face a problem with the idea that the deformations (5.50) for $s \geq 1$ induce the exchange between equivalent irreps of $SO(3,2)$. Starting with the irreps $D(s+1,s)$, $s \geq 1$, their equivalent irreps are $D(2-s,s)$. These, not only they fall below the unitarity bound $\Delta \geq s+1$, but for $s > 2$ they appear to correspond to operators with negative scaling dimensions. Clearly, such irreps cannot represent physical fields in a QFT. This is to be contrasted with the case of the irreps $D(1,0)$ and $D(2,0)$ which are both above the unitarity bound.

The remedy of the above problem is suggested by old studies of three- dimensional gauge theories. In particular, it is well-known that to a three-dimensional gauge potential $A_i(p)$ corresponds (we work in momentum space for simplicity), a conserved current $J_i(x) \propto i\epsilon_{ikj}p_j A_k(p)$, while to the three-dimensional spin-2 gauge potential $g_{ij}(p)$ corresponds a conserved current (related to the Cotton tensor [22]), $T_{ij}(p) \propto \Pi^{(1.5)}_{ijkl}g_{kl}(p)$ (see below for the definitions). A similar construction associates to each gauge field belonging to the irrep $D(2-s,s)$ a physical current in the irrep $D(s+1,s)$. Therefore, we need actually two steps to understand the effect of the double-trace deformations (5.50); firstly the deformation will produce what it looks like a correlator for an operator transforming under $D(2-s,s)$ that may be viewed as a gauge field, secondly we transform this correlator to one of a conserved current $D(s+1,s)$. Moreover, these conserved currents have opposite parity from the gauge fields, so we expect that parity plays a role in our discussion.

Let us be more concrete and see what all the above mean in practice. Consider a boundary CFT with a conserved current J_i having momentum space 2-pt function

$$\langle \mathcal{J}_i \mathcal{J}_i \rangle_0 \equiv (\mathcal{J}_0)_{ij} = \tau_1 \frac{1}{|p|}\Pi_{ij} + \tau_2 \varepsilon_{ijk}p_k\,, \quad \Pi_{ij} \equiv p_i p_j - \delta_{ij}p^2\,. \tag{5.51}$$

Consider the irrelevant double-trace deformation

$$\frac{f_1}{2}\int \mathcal{J}_i \mathcal{J}_i\,, \tag{5.52}$$

and calculate

$$(\mathcal{J}_{f_1})_{ij} \equiv \langle \mathcal{J}_i \mathcal{J}_i \, e^{\frac{f_1}{2}\int \mathcal{J}\mathcal{J}} \rangle = (\mathcal{J}_0)_{ij} + \frac{f_1}{2} \int \langle \mathcal{J}_i \mathcal{J}_j \Big| \mathcal{J}_k \mathcal{J}_k \rangle + \ldots \tag{5.53}$$

Now assume: i) large-N expansion $\mathcal{J}_i\mathcal{J}_j \sim (\mathcal{J})_{ij} + O(1/N)$ and ii) existence of a UV fixed-point. The leading-N resummation yields

$$f_1^2(\mathcal{J}_{f_1})_{ij} = \hat{\tau}_1 \frac{1}{|p|} \Pi_{ij} + \hat{\tau}_2 \varepsilon_{ijk} p_k \tag{5.54}$$

$$\hat{\tau}_1 \simeq \frac{f_1}{|p|} + \frac{1}{|p|^2} \frac{\tau_1}{\tau_1^2 + \tau_2^2} + \ldots \tag{5.55}$$

$$\hat{\tau}_2 \simeq -\frac{1}{|p|^2} \frac{\tau_2}{\tau_1^2 + \tau_2^2} + \ldots \tag{5.56}$$

Dropping the non-conformally invariant term $f_1/|p|$ we get

$$f_1^2(\mathcal{J}_{f_1})_{ij} = \frac{\tau_1}{\tau_1^2 + \tau_2^2} \frac{1}{|p|^3} \Pi_{ij} - \frac{\tau_2}{\tau_1^2 + \tau_2^2} \varepsilon_{ijk} p_k \ . \tag{5.57}$$

This is the 2-pt function of a conformal operator $\hat{A}_i(\bar{p})$ transforming in the irrep $D(1,1)$. It lies below the unitary bound $\Delta \geq s+1$ of $SO(3,2)$, therefore it must be a gauge field. Define then the current $\hat{\mathcal{J}}_i = i\varepsilon_{ijk} p_j \hat{A}_k$ that has 2-pt function

$$\langle \hat{\mathcal{J}}_i \hat{\mathcal{J}}_j \rangle = \frac{\tau_1}{\tau_1^2 + \tau_2^2} \frac{1}{|p|^3} \Pi_{ij} - \frac{\tau_2}{\tau_1^2 + \tau_2^2} \varepsilon_{ijk} p_k \ . \tag{5.58}$$

It follows that there exists a "dual" theory with current $\hat{\mathcal{J}}_i$ that has 2-pt function obtained from (5.51) by

$$\tau \to -\frac{1}{\tau}\,, \quad \tau = \tau_2 + i\tau_1 \ . \tag{5.59}$$

Similarly, we may consider a boundary CFT having an energy momentum tensor with 2-pt function

$$\langle T_{ij} T_{kl} \rangle = \kappa_1 \frac{1}{|p|} \Pi^{(2)}_{ij,kl} - \kappa_2 \Pi^{(1.5)}_{ij,kl} \ , \tag{5.60}$$

where

$$\begin{aligned} \Pi^{(2)}_{ij,kl} &= \frac{1}{2}\Big[\Pi_{ik}\Pi_{jl} + \Pi_{il}\Pi_{jk} - \Pi_{ij}\Pi_{kl}\Big] \ , \\ \Pi^{(1.5)}_{ij,kl} &= \frac{1}{4}\Big[\varepsilon_{ikp}\Pi_{jl} + \varepsilon_{jkp}\Pi_{il} + \varepsilon_{ilp}\Pi_{jk} + \varepsilon_{jlp}\Pi_{ik}\Big] \ . \end{aligned} \tag{5.61}$$

The boundary irrelevant "double-trace" deformation

$$\frac{f_2}{2} \int T_{ij} T_{ij} \ , \tag{5.62}$$

leads (under the same large-N, existence of UV fixed point assumptions as above), to a theory with an energy momentum tensor that has 2-pt function obtained from (5.60) by [23]

$$\kappa \to -\frac{1}{\kappa}\ , \quad \kappa = \kappa_2 + i\kappa_1\ . \tag{5.63}$$

It is not difficult to imagine that the picture above generalizes to all higher-spin currents in a three-dimensional CFT.

Now let us discuss what all the above boundary properties mean for the bulk HS gauge theory. The first thing to notice is of course that the form of the transformations (5.59) and (5.63) is reminiscent of S-duality transformations. Then, we must ask what could be the bulk action that yields (5.51) and (5.60). For example, (5.51) may be the on-shell boundary value of the $U(1)$ action with a θ-term on AdS$_4$

$$I = \frac{1}{8\pi}\int d^4x\sqrt{g}(\frac{4\pi}{e^2}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{2\pi}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})\ . \tag{5.64}$$

We will show now that the bulk dual of the double-trace boundary deformation discussed previously is a canonical duality transformation i.e. a canonical transformation that interchanges the bulk canonical variables. We use the ADM form of the Euclidean AdS$_4$ metric (with radius set to 1)

$$ds^2 = d\rho^2 + \gamma_{ij}dx^i dx^j\ , \quad \gamma_{ij} = e^{2\rho}\eta_{ij}\ , \ \gamma = \det\gamma_{ij}\ , \ \ i,j = 1,2,3\ . \tag{5.65}$$

to write the bulk action (5.64) in terms of the canonical variables as

$$\begin{aligned} I &= \int d\rho\, d^3x\, \sqrt{\gamma}\left[\Pi^i\dot{A}_i - \mathcal{H}(\Pi^i, A_i)\right]\ , & (5.66)\\ \mathcal{H}(\Pi^1, A_i) &= \frac{1}{e^2}\gamma^{-1}\gamma_{ij}\left(\mathcal{E}^i\mathcal{E}^j - \mathcal{B}^i\mathcal{B}^j\right)\ , & (5.67)\\ \sqrt{\gamma}\Pi^i &= \frac{2}{e^2}\mathcal{E}^i + i\frac{\theta}{4\pi^2}\mathcal{B}^i\ , \ \ \mathcal{E}^i = \sqrt{\gamma}E^i\ , \ \mathcal{B}^i = \sqrt{g}B^i\ , & (5.68)\end{aligned}$$

with $E^i = F^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ the usual electric and magnetic fields. The essence of the Hamilton-Jacobi approach to AdS holography is that for a given ρ_0 the variation of the bulk action with respect to the canonical variable A_i gives, on shell, the canonical momentum Π^i at ρ_0. For $\rho_0 \to \infty$ the latter is interpreted as the *regularized* 1-pt function in the presence of sources, the reason being that requiring the regularity of the classical solutions inside AdS gives a relation between Π^i and A_i. Finally, to reach the boundary one invokes a further technical step, (sometimes called holographic renormalization), such as to obtain finite 1-pt functions from which all correlation functions of the boundary CFT can be found. Schematically we have

$$\frac{1}{\sqrt{\gamma}}\left.\frac{\delta I}{\delta A_i(\rho_0, x_i)}\right|_{on-shell} = \Pi^i(\rho_0, x_i) \sim_{\rho_0\to\infty} \langle \mathcal{J}^i(x_i)\rangle_{A_i} \tag{5.69}$$

In fact, one can show that for a $U(1)$ field on AdS_4 there is no need for renormalization as the solutions of the bulk e.o.m. give finite contributions at the boundary.

Next we consider canonical transformations in the bulk to move from the set variables (A_i, Π^i) to the new set $(\tilde{A}_i, \tilde{\Pi}^i)$. In particular, we may consider a generating functional of the 1st kind (see e.g. [25]) of the form

$$\mathcal{F}[A_i, \tilde{A}_i] = \frac{1}{2}\int_{\rho=\,fixed} d^3x\ \sqrt{\gamma} A_i(\rho, x_i)\epsilon^{ijk}\tilde{F}_{jk}(\rho, x_i)\,. \tag{5.70}$$

This induces the transformations

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta A_i} \equiv \Pi^i = \tilde{B}^i\,, \tag{5.71}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta \tilde{A}_i} \equiv -\tilde{\Pi}^i = B^i\,. \tag{5.72}$$

For $\theta = 0$ these are the standard duality transformations $E^i \to B^i$, $B^i \to -E^i$. We have at our disposal now two bulk actions, one written in terms of (A_i, Π^i) and the other in terms of $(\tilde{A}_i, \tilde{\Pi}^i)$, which according to (5.69) give at $\rho = \infty$,

$$\langle \mathcal{J}_i\rangle_{A_i} = \tilde{B}_i = i\epsilon_{ijk}p_j\tilde{A}_k\,, \tag{5.73}$$

$$\langle \tilde{\mathcal{J}}_i\rangle_{\tilde{A}_i} = -B_i = -i\epsilon_{ijk}p_jA_k\,. \tag{5.74}$$

From the above 1-pt functions we can calculate the corresponding 2-pt functions by functionally differentiating with respect to A_i and $\tilde{A}_i$. We now take the following ansatz for the the matrix $\delta\tilde{A}_i/\delta A_j$

$$\frac{\delta\tilde{A}_i}{\delta A_j} = C_1\frac{1}{p^2}\Pi_{ij} + C_2\epsilon_{ijk}\frac{p_k}{|p|} + (\xi - 1)\frac{p_ip_j}{p^2}\,, \tag{5.75}$$

where ξ plays as usual the role of gauge fixing, necessary for its inversion. Then from (5.73) and (5.74) we find, independently of ξ

$$\langle \mathcal{J}_i\mathcal{J}_k\rangle\langle\tilde{\mathcal{J}}_k\tilde{\mathcal{J}}_j\rangle = -\Pi_{ij}\,, \tag{5.76}$$

with Π_{ij} defined in (5.51). It is then easy to verify that if $\langle \mathcal{J}_i\mathcal{J}_k\rangle$ is given by the r.h.s. of (5.51), $\langle\tilde{\mathcal{J}}_k\tilde{\mathcal{J}}_j\rangle$ is given by the r.h.s. of (5.58). In other words, we have shown that the bulk canonical transformation generated by (5.70) induces the S-transformation (5.59) on the boundary 2-pt functions. We expect that our result generalizes to linearized bulk gravity in the Hamiltonian formalism[4] and linearized higher-spin theories.

[4]See [26] for a recent discussion of the duality in this context.

An intriguing property of the above transformations generated by the boundary double-trace deformations is that combined with the trivial transformation defined as

$$\tau \to \tau + 1 \, , \tag{5.77}$$

form the SL(2,Z) group [24]. The transformation (5.77) is the boundary image of the bulk shift of the θ-angle

$$\theta \to \theta + 2\pi \, . \tag{5.78}$$

We expect that an analogous effect takes place in linearized higher-spin gauge theories on AdS_4 when the appropriate θ-terms are introduced in the bulk [23, 27]. The above suggest a special role for the SL(2,Z) group in the study of HS gauge theories, even at the quantum level.

6 Discussion

It has been suggested [28] that HS gauge theories emerge at the tensionless limit of string theory, in a way similar to the emergence of supergravity at the limit of infinite string tension. It would be extremely interesting to quantify the above statement. A step in this direction is the study of the holography of higher-spin theories on AdS spaces. In this direction, both the study of specific models, as well as investigations of generic holographic properties of HS theories are important. The three-dimensional $O(N)$ vector model provides a concrete example of a theory with a holographic HS dual. On the other hand, bulk dualities of linearized higher-spin theories may have far reaching consequences for their CFT duals. For example, if a theory possesses a HS dual with a self-duality property, its boundary double-trace deformations despite being irrelevant might lead to a well-defined UV completion of the theory. Also, it is conceivable that the self-duality property of linearized HS theories is a remnant of a string theory duality in the tensionless limit. Finally, it is well-known that 2-pt functions of three-dimensional spin-1 conserved currents can describe observable properties of Quantum Hall systems [29]. It is then interesting to ask whether boundary correlation functions of higher-spin currents may describe observables properties of physical systems. In particular, it is intriguing to suggest [30] that linear gravity in AdS_4 may correspond to special kinds of three-dimensional fluids in which SL(2,Z) or a subgroup of it play a role.

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